ON QUADRATIC AND HIGHER NORMALITY OF SMALL CODIMENSION PROJECTIVE VARIETIES

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ABSTRACT. Ran proved that smooth codimension 2 varieties in $\mathbf{P}^{m+2}$ are j-normal if $(j+1)(3j-1) \leq m-1$, in this paper we extend this result to small codimension projective varieties. Let X be a r codimension subvariety of $\mathbf{P}^{m+r}$, we prove that if the set $\Sigma_{(j+1)}$ of $(j+1)$-secants to X through a generic external point is not empty, $2(r+1)j \leq m-r$ and $(j+1)((r+1)j-1) \leq m-1$ then X is j-normal. If X is given by the zero locus of a section of a rank r vector bundle E on $\mathbf{P}^{m+r}$, we prove that deg $\Sigma_{j+1} = \frac{1}{(j+1)!}\prod_{i=0}^{j} c_r(E(-i))$. Moreover we get a new simple proof of Zak's theorem on linear normality if $m \geq 3r$. Finally we prove that if $c_r(N(-2)) \neq 0$ and $6r \leq m-4$ then X is 2-normal.

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1. INTRODUCTION

A variety $X \subset \mathbf{P}^n$ is called j-normal if the restriction map $H^0(\mathbf{P}^n, \mathcal{O}(j)) \longrightarrow H^0(X, \mathcal{O}(j))$ is surjective. Hartshorne's conjecture [9] implies that smooth varieties $X \subset \mathbf{P}^n$ of small codimension are j-normal. Peternell, Le Potier , Schneider [13] and Ein [3] proved indipendently that smooth codimension 2 varieties $X \subset \mathbf{P}^n$ are 2-normal if $n \geq 10$. This bound is probably not sharp (Hartshorne's conjecture implies $n \geq 6$) but it is interesting because it does not depend on the degree of X (for similar bounds depending on the degree, see [8] [11]).Ein's results were extended to higher normality by Alzati and Ottaviani in [1], but the techniques of those papers seem not to work in codimension ≥ 3 because the Koszul complexes appearing in the proof have greater length and are difficult to control. On the other hand Ran in [16] proved, with different techniques, that smooth codimension 2 varieties $X \subset \mathbf{P}^n$ are j-normal if $n \geq 3j^2+2j+2$. Ran constructs explicitly, for any $Y \in H^0(X, \mathcal{O}(j))$, a hypersurface F in $\mathbf{P}^n$ of degree j as the union of lines which intersects Y with multiplicity $\geq j+1$. This works because the assumption implies that the locus of $j+1$-secants is not empty. In our doctoral thesis, we expanded all the details of

1991 *Mathematics Subject Classification.* 14M07(primary),14N05,14N10(secondary).

Ran's paper and we were able to prove the following theorem which gives bounds for j-normality also in codimension $r \geq 3$.

Denote by $\Sigma_{(j+1)}$ the set of $(j+1)$-secants to X through a (generic) external point.

Theorem 1.1. *Let X be a r codimension subvariety of $\mathbf{P}^{m+r}$; if*

$$\Sigma_{(j+1)} \neq \emptyset$$

$$2(r+1)j \leq m-r \qquad \text{and} \qquad (j+1)((r+1)j-1) \leq m-1$$

then:

$$\rho_j : H^0(\mathbf{P}^{m+r}, \mathcal{O}_{\mathbf{P}^{m+r}}(j)) \longrightarrow H^0(X, \mathcal{O}_X(j))$$

is surjective.

If $r=2$, the numerical assumptions of theorem 1.1 are exactly as in [16], while Ran is able to show that in this bound if $\Sigma_{j+1} = \emptyset$ then X is a complete intersection.

Ran himself pointed out in a remark at the end of the paper that his proof could also be extended to higher codimension. When X is the zero locus of a section of a vector bundle, then the numeric assumption is more explicit.

Theorem 1.2. *Let X be a m dimension variety in $\mathbf{P}^{m+r}$ given by the zero locus of a section of a rank r vector bundle E on $\mathbf{P}^{m+r}$. We have*

$$deg\ \Sigma_{j+1} = \frac{1}{(j+1)!}\prod_{i=0}^{j} c_r(E(-i))$$

Corollary 1.3. *With the assumptions of the theorem 1.2, if*

$$c_r(E(-i)) \neq 0 \quad \forall i = 1 \dots j$$

$$2(r+1)j \leq m-r \qquad \text{and} \qquad (j+1)((r+1)j-1) \leq m-1$$

then:

$$\rho_j : H^0(\mathbf{P}^{m+r}, \mathcal{O}_{\mathbf{P}^{m+r}}(j)) \longrightarrow H^0(X, \mathcal{O}_X(j))$$

is surjective.

In section 4 we get a new proof of Zak theorem about linear normality with the assumption $n \geq 4r$. In the same range there is still another proof due to Faltings [4]. Moreover in this paper we prove the following result on quadratical normality where the numeric assumption is easier checked. This is a partial answer to problem 12 in Schneider list [17].

Theorem 1.4. *Let X be a m dimension variety in $\mathbf{P}^{m+r}$. If*

$$c_r(N(-2)) \neq 0 \quad \text{and} \quad 6r \leq m-4$$

then X is 2-normal.

I thank G.Ottaviani for the precious suggestions and the useful discussions and L.Göttsche for some ideas used for the proof of theorem 1.2.

2. Proof of theorem 1.1

Consider a branched covering that is a finite surjective morphism between two irreducible and nonsingular algebraic varieties V and W $f : V \to W$; let d be the degree of f. As we are assuming that V and W are non-singular, f is flat and consequently the direct image $f_*\mathcal{O}_V$ is locally free of rank d on W. The trace $Tr_{V/W} : f_*\mathcal{O}_V \to \mathcal{O}_W$ gives rise to a splitting: $f_*\mathcal{O}_X = \mathcal{O}_W \oplus F$, where $F = ker(Tr_{V/W})$. We shall be concerned with the rank $d-1$ vector bundle on W: $E = F^*$. E will be termed *vector bundle associated with the covering* f. Let $e_f(x) = dim_{\mathbf{C}}(\mathcal{O}_x X/f^* m_{f(V)})$ be the local degree of f in x which counts the number of sheets of covering that come together at x.

Theorem 2.1 (Gaffney-Lazarsfeld)**.** *Let V and W be varieties of dimension n and $f : V \longrightarrow W$ a branched covering of degree d; if the vector bundle associated with a branched covering is ample, then there exists at least one point $x \in V$ at which*

$$e_f(x) \geq min(d, n+1).$$

Proof See [6]. Lazarsfeld himself points out that smoothness of W is not essential.

Thanks to this theorem, we are able to prove the following Lemma:

Lemma 2.2. *Let X be a r-codimensional subvariety of $\mathbf{P}^n$, if $r \cdot k \geq n$ and the set of k-secant lines to X through an external point P is not empty, then there exists at least a k-secant through this point at which the k points coincide.*

Proof We consider the projection from P of k-secants on a generic hyperplane $\mathbf{P}^{n-1}$; let f be its restriction to the points of X, and Y the image of f. $X' = f^{-1}(Y)$, X' is the set of points in X lying on a k-secant. The dimension n' of X' and Y is $n-1-k(r-1)$ and $f : X' \longrightarrow Y$ is a finite covering with degree k: by our assumptions the degree of the covering is less than or equal to $n'+1$. If we prove that the vector bundle associated with the covering is ample, then we can use the theorem of Gaffney and Lazarsfeld to prove that there exists a point at which the sheets of covering come together. We denote C as the cone of k-secants through an external point P; since there are k points of X' for each k-secant, we observe that X' is a divisor of C and since the point P is external to X this divisor is disjoint from singularities of C. Let C' be the desingularization of C, we have:

$$C' = \mathbf{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(1)),$$

then X' is isomorphic to a divisor of C'. $f_*\mathcal{O}_{X'}$ is a vector bundle of rank k; we want to prove that:

$$f_*\mathcal{O}_{X'} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \ldots \oplus \mathcal{O}_Y(1-k).$$

X' is the zero locus of a section of $\mathcal{O}_{\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(1))}(k)$; in fact, from [10] we have: $Pic(C') = Pic(Y) \oplus \mathbb{Z}H$, where H is hyperplane section. X' is a divisor which meets the generic fibre in k points and it is disjoint to the infinite section, and so X is linearly equivalent to kH. Now we consider the associated exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-k) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

Let π be the projection from $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$ to Y; applying π_* to the sequence we obtain:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \pi_* \mathcal{O}_{X'} \longrightarrow R^1\pi_*\mathcal{O}_{\mathbf{P}}(-k) \longrightarrow 0.$$

Using the exercise 8.4 of [10], (page 253) we prove that:

$$R^1\pi_*\mathcal{O}(-k) \simeq \pi_*(\mathcal{O}(k-2))^* \otimes \mathcal{O}_Y(-1)$$

and from the same exercise we have:

$$\pi_*\mathcal{O}(k-2) \simeq S^{k-2}(\mathcal{O} \oplus \mathcal{O}(1)) = \mathcal{O} \oplus \mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(k-2)$$

then

$$R^1\pi_*\mathcal{O}(-k) = \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \ldots \mathcal{O}(-k+1)$$

substituting in the exact sequence we get:

$$\pi_*\mathcal{O}_{X'} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-2) \oplus \ldots \mathcal{O}_Y(1-k)$$

then

$$\pi_*\mathcal{O}_{X'} = \mathcal{O}_Y \oplus F$$
$$\pi_*\mathcal{O}_{X'} = f_*\mathcal{O}_{X'}$$

where F is a vector bundle whose dual is ample. We can now use the Gaffney-Lazarsfeld's theorem to obtain the thesis.

Lemma 2.3. *Let G be a generic hypersurface of $\mathbf{P}^n$ of degree j passing through a point P, then the variety of lines through P lying in G is a complete intersection of $\mathbf{P}^{n-2}$ with dimension $n-j-1$ and degree $j!$.*

Proof. We can choose a coordinate system such that P is the point $(a, 0, 0, \ldots, 0)$. Let π be the hyperplane $x_0 = 0$; for every point Q of π we consider the line r through P and Q that is $(a(1-t), tx_1, \ldots, tx_n)$. G is given by $F(y_0, \ldots, y_n) = 0$ with $F(y_0, \ldots, y_n) = by_0^j + f_1(y_1, \ldots, y_n)y_0^{j-1} + \ldots + f_j(y_1, \ldots, y_n)$ where f_i are polynomials of degree i; since $P \in G$ we have $b = 0$. A line r lie on G if and only if:

$$F(a(1-t), tx_1, \ldots, tx_n) = ty_0^{j-1}f_1(x_1, \ldots x_n) + \ldots t^j f_j(x_1, \ldots, x_n) = 0$$

for every t, and so we must have: $f_i(x_1, \ldots, x_n) = 0 \quad \forall i = 1, \ldots, n$. Since G is generic and f_1 is linear, this gives a transversal intersection contained in $\mathbf{P}^{n-2}$. Finally we get that the variety of lines of G through a point $P \in G$ is a complete intersection of degree $j!$ and dimension $n-1-j$.

Let X be a subvariety of $\mathbf{P}^{m+r}$; we denote by Σ_j the cycle of j-secant lines to X through an external point.

Proof of theorem 1.1 Consider a generic element Y of the linear system $\mid \mathcal{O}_X(j) \mid$. Since the locus of $(j+1)$-secants through a generic point is not empty, then X can not be included in a hypersurface of degree j and so $H^0(\mathcal{I}_X(j)) = 0$. In order to prove the theorem we just have to find one hypersurface of degree $\leq j$ which contains Y.

We define R^k=$\{(y,z) \in Y \times \mathbf{P}^{m+r}$: $\exists$ a line L from $z \in \mathbf{P}^{m+r}$ such that $L \cap Y$ has multiplicity $\geq k$ in $y\}$. Let p and q the projections of R^k to Y and to $\mathbf{P}^{m+r}$ respectively:

$${}_zR^k = p(q^{-1}(z)) \qquad R^k_y = q(p^{-1}(y))$$

R^k_y is the set of points on lines from y intersecting Y with multiplicity $\geq k$ and it is a cone of vertex y. In a neighborhood of y we can identify $\mathbf{P}^{m+r}$ with $\mathbf{C}^{m+r}$ where y

is the origin, Y is defined in an appropriate neighborhood of y by $(r+1)$ polynomials $f_1 \dots f_{r+1}$. R^k_y is given by vanishing of the homogeneous components of degree $\leq k-1$; and so if a generic line L of $\mathbf{P}^{m+r}$ meets R^k_y in k points, then $L \in R^k_y$. Moreover:

$$\dim R^k_y \geq m+r-(r+1)(k-1) \forall y \in Y.$$

Let $F = q(R^{j+1})$ be the set of points of $\mathbf{P}^{m+r}$ on lines which intersect Y with multiplicity $\geq j+1$ in one point: we want to prove that F is the hypersurface we looked for.
$Y \subset F$ because dim $R^{j+1}_y \geq 0$ $\forall y \in Y$. The first step is to prove that

$$F \underset{\neq}{\subseteq} \mathbf{P}^{m+r}.$$

Let $Y' = X \cdot G$ where G is a generic hypersurface of degree j. Y' is obtained by Y by semicontinuity, so the dimension of F passing from Y to Y' cannot decrease, and since the (j+1)-secants to Y' are contained in G we obtain: $\dim F \leq m+r-1$.

Next step is to prove that :

$$\dim F \geq m+r-1.$$

The set of (j+1)-secants to $Y' = X \cdot G$ through an external point $P \in G$ is given by the intersection of Σ_{j+1} with the variety of Lemma 2.3, and so, by the assumption, we obtain a variety with degree different to 0; j times this degree gives the virtual degree of $(j+1)$ secants intersecting a generic line of $\mathbf{P}^{m+r}$. Since Y is a degeneration of Y', this virtual degree is the same and it is different from 0 as stated previously.
Let B the locus of $(j+1)$-secants to Y interecting a generic line, B has dimension ≥ 0 in the grassnammian of lines in $\mathbf{P}^{m+r}$ and it is given by $A \cap S$ where:

$$A = \{\text{lines of } \mathbf{P}^{m+r} \text{ that are } (j+1)\text{-secant to } Y\}$$

$$S = \{\text{lines of } \mathbf{P}^{m+r} \text{ intersecting a given line}\}$$

$$\dim \{A \cap S\} \geq 0 \Longrightarrow \operatorname{codim}\{A \cap S\} \leq 2(m+r-1).$$

Since the line is generic, we have:

$$\operatorname{codim}\{A \cup S\} = \text{codim A} + \text{ codim S}$$

$$\text{codim S} = m+r-2 \Longrightarrow \operatorname{codim} A \leq m+r$$

$$\dim A \geq m+r-2$$

. Let A' be the variety of points of A, then we have:

$$\dim A' \geq m+r-1.$$

Now we have to prove that $A' = F$.
The inclusion $F \subset A'$ is trivial; we want to prove that if $p \in A'$, then $p \in F$. From Lemma 2.2 we have that if p lies on a $(j+1)$-secant to Y then it lies also on a line intersecting Y with multiplicity $(j+1)$ in a point of Y. Finally we have to prove that

$$deg F \leq j.$$

Let suppose that a generic line L of $\mathbf{P}^{m+r}$ meets F in (j+1) points $z_1 \dots z_{j+1} \in L \cap F$. Let's compute c_i= codim $({}_{z_i}R^{j+1}, Y)$:

$$\dim\ R^{j+1} = \dim\ Y + \dim\ p^{-1}(y) = \dim F + \dim\ q^{-1}(z),$$

since dim $p^{-1}(y)$ = dim R_y^{j+1} and dim $q^{-1}(z)$ =dim ${}_zR^{j+1}$, as previously stated, we have:

$$\dim R^{j+1} \geq 2m - 1 + r - (r+1)j$$

then

$$\dim {}_{z_i}R^{j+1} \geq m - (r+1)j$$

$$c_i = \operatorname{codim}\,({}_{z_i}R^{j+1}, Y) \leq (r+1)j - 1$$

By the Lefschetz-Barth's theorem and by the assumption. we have

$$\mathbf{C} = H^{2c_i}(\mathbf{P}^{m+r}, \mathbf{C}) = H^{2c_i}(Y, \mathbf{C}) \Longrightarrow \bigcap_{i=1}^{j+1} {}_{z_i}R^{j+1} \neq \emptyset$$

in fact:

$$2c_i \leq m - r - 2 \quad e \quad (j+1)((r+1)j - 1) \leq m - 1.$$

Let $y \in \bigcap_{i=1}^{j+1} {}_{z_i}R^{j+1}$ then $z_i \in L \cap R_y^{j+1}$ *per* $i = 1 \dots j+1$ and so $L \subset R_y^{j+1}$. This is a contradiction as L is generic. We deduce that $degF \leq j$.

3. Proof of theorem 1.2

Proof of theorem 1.2

Let P be the fixed point and $Q \subset G(\mathbf{P}^1, \mathbf{P}^n)$ the space of lines from P, $Q \simeq \mathbf{P}^{n-1}$; let:

$$T = \{(q, l) \mid q \in \mathbf{P}^n \quad l \in Q \quad q \in l\}$$

and α and β be the projections of T on $\mathbf{P}^n$ and Q respectively.
T is a $\mathbf{P}^1$-bundle on Q and the fibre is given by all the points lying on lines l, we can view it as the projectivised of $\mathcal{O}_Q \oplus \mathcal{O}_Q(-1)$.
Let $(T/Q)^{k+1}$ be the $(k+1)$-power of fibre of T on Q,that is:

$$(T/Q)^{k+1} = \underbrace{T \times_Q T \times_Q \dots \times_Q T}_{k+1 times}$$

We call $Z \in (T/Q)^{k+2}$ the incidence variety in $T \times_Q (T/Q)^{k+1}$, that is:

$$Z = \{(x_0, \dots, x_{k+1} \in (T/Q)^{k+2} \quad \mid x_0 = x_i \text{ for same } i \in (1, \dots, k+1)\}$$

Let p and q be the projections of Z on T and $(T/Q)^{k+1}$ respectively; we denote:

$$E^{(k+1)} = q_*(p^*\alpha^*(E))$$

$E^{(k+1)}$ is a vector bundle on $(T/Q)^{k+1}$ of rank $r(k+1)$. Let s be the section of E such that X is the zero locus of s; $s^{(k+1)} = q_*(p^*\alpha^*(s))$ is a section of $E^{(k+1)}$ which vanishes in the set: $\{(x_1, \dots, x_{k+1} \in (T/Q)^{k+1} \quad \mid \alpha(x_i) \in X\}$. The line through $\alpha(x_1), \dots, \alpha(x_{k+1})$ is a $(k+1)$-secant to X. Considering that the rearrangement of those points gives the same $(k+1)$-secant to X, from Portous' formula we have that, if the dimension is zero, the number of $(k+1)$-secant is given by the degree of the top Chern-class $c_{(k+1)r}(E^{(k+1)})$ divided by $(k+1)!$. So if we want to know the degree of $(k+1)$-secants we have to compute $c_{(k+1)}(E^{(k+1)})$. Let $q_1, \dots q_{k+1}$ be the projections of $(T/Q)^{k+1}$ on $\mathbf{P}^n$; we have the following exact sequence:

$$0 \longrightarrow q^*E \otimes \mathcal{O}(-\Delta_{1,k+1} \dots - \Delta_{k,k+1}) \longrightarrow E^{(k+1)} \longrightarrow E^{(k)} \longrightarrow 0$$

with $\Delta_{i,j} = \{(x_1, \dots x_{k+1} \in (T/Q)^{k+1} \quad \mid x_i = x_j\}$. From sequence we have:

$$c_{(k+1)r}E^{(k+1)} = c_{kr}E^{(k)}c_r(q^*E \otimes \mathcal{O}(-\Delta_{1,k+1} \dots - \Delta_{k,k+1})).$$

It is necessary to determine the cohomology of T and $(T/Q)^{k+1}$.
Let α and β be the projections of T on $\mathbf{P}^n$ and Q respectively: T is blow-up of $\mathbf{P}^n$ in P; we call D the exceptional divisor and $H = \alpha^*(\mathcal{O}_{\mathbf{P}^n}(1))$, then we have: $H - D = \beta^*(\mathcal{O}_Q(1)), D = \mathcal{O}_T(1)$. The Wu-Chern's equation gives: $D^2 + \beta^*\mathcal{O}_Q(1)D = 0$.
The intersection ring of T is generated by two elements:

$$\langle D, H - D\rangle = \langle D, \beta^*\mathcal{O}_Q(1)\rangle.$$

For the next degrees we have:

$$(\beta^*\mathcal{O}_Q(1))^2 D = -\beta^*\mathcal{O}_Q(1)D^2 = D^3$$

$$\vdots$$

$$(\beta^*\mathcal{O}_Q(1))^n D = (-1)^{n-1}\beta^*\mathcal{O}_Q(1)D^n = D^{n+1}.$$

We observe that $H^n = 1$ and $D^n = (-1)^{n-1}$ in fact $D_{|D} = \mathcal{O}_D(-1)$ and $D \simeq \mathbf{P}^{n-1}$.
Consider now the fibred product $T \times_Q T$: $H^*(T)$ is generated by D as $H^*(Q)$-module; $H^*(T \times_Q T) = H^*(T) \times_{H^*(Q)} H^*(T)$ is generated by $D \otimes 1 = D_1$ and $1 \otimes D = D_2$ as $H^*(Q)$-module; if we consider it as a vector space we have:

$$H^2(T \times_Q T) = \langle D_1, D_2, \beta^*\mathcal{O}_Q(1)\rangle$$

$$H^4(T \times_Q T) = \langle D_1^2, D_2^2, (\beta^*\mathcal{O}_Q(1))^2, D_1D_2\rangle$$

$$\vdots$$

$$H^{2j}(T \times_Q T) = \langle D_1^j, D_2^j, (\beta^*\mathcal{O}_Q(1))^j, D_1D_2\beta^*\mathcal{O}_Q(1)^{j-2}\rangle$$

we denote: $H_1 = q_1^*(H) \quad H_2 = q_2^*(H)$

$$H_1 = D_1 + \beta^*\mathcal{O}_Q(1) \quad H_2 = D_2 + \beta^*\mathcal{O}_Q(1).$$

We prove that: $\Delta_{1,2} = D_1 + D_2 + \beta^*\mathcal{O}_Q(1)$. Let p_1 and p_2 be the projection of $T \times_Q T$ on the two factors. $\Delta_{1,2}$ is given by the zero locus of a section of $p_1^*\mathcal{O}(1) \otimes p^*Q_{rel}$ (see [14],page 242). We know that $c_1(p_1^*\mathcal{O}(1)) = D_1$; consider now the exact section

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \beta^*\mathcal{O} \oplus \beta^*\mathcal{O}(1) \longrightarrow Q_{rel} \longrightarrow 0.$$

We get $c_1(p_2^*Q_{rel}) = D_2 + \beta^*\mathcal{O}(1)$ and so we have $\Delta_{1,2} = D_1 + D_2 + \beta^*\mathcal{O}_Q(1)$.

For the general case we have that:
$H^2(T/Q)^{k+1}$ is generated by $D_1, D_2, \dots D_{k+1}, \beta^*\mathcal{O}_Q(1)$,
$\vdots$
$H^{2m}(T/Q^{k+1})$ is generated by $(\beta^*\mathcal{O}_Q(1))^m, D_{i_1}\dots D_{i_t}\beta^*\mathcal{O}_Q(1)^{m-t}$. Moreover: $\Delta_{i,j} = D_i + D_j + \beta^*\mathcal{O}_Q(1)$.

Now we prove the theorem proceeding by induction on k: for $k = 1$ the exact sequence is:

$$0 \longrightarrow q_2^*(E) \otimes O(-\Delta_{1,2}) \longrightarrow E^{(2)} \longrightarrow q_1^*(E) \longrightarrow 0$$

$$\begin{aligned} c_{2r}(E)^{(2)} &= c_r((q_1(E))c_r(q_2^*(E) \otimes O(-D_1 - D_2 - \beta^*\mathcal{O}_Q(1)) \\ &= c_r(E)H_1^r[c_r(E)H_2^r + c_{r-1}(E)H_2^{r-1}(-D_1 - D_2 - \beta^*\mathcal{O}_Q(1)) + \\ &\quad \dots + c_{r-i}(E)H_2^{r-i}(-D_1 - D_2 - \beta^*\mathcal{O}_Q(1))^i + \\ &\quad \dots + (-D_1 - D_2 - \beta^*\mathcal{O}_Q(1))^r] \end{aligned}$$

since: $H_1 D_1 = 0$ and $D_2 + \beta^* \mathcal{O}_Q(1) = H_2$ we have: $c_{2r} = c_r(E)c_r(-1)H_1^r H_2^r$. Now we suppose the statement true for $n \geq k$ and we try to prove it for $n = k+1$.

$$\begin{aligned} c_{(k+1)r} &= c_{kr} E^{(k)} c_r(q_{k+1}^*(E) \otimes O(-D_1 - D_2 \ldots - D_k - kD_{k+1} - k\beta^* \mathcal{O}_Q(1)) \\ &= c_r(E)c_r(E(-1)) \ldots c_r(E(-k+1)) H_1^r \ldots H_{k-1}^r [c_r(E) H_{k+1}^r + \\ & \quad c_{r-1}(E) H_{k+1}^{r-1}(-D_1 - D_2 \ldots) + \ldots (-D_1 \ldots - k\beta^* \mathcal{O}_Q(1))^r] \end{aligned}$$

since we know that: $H_i D_i = 0$ and $D_{k+1} + \beta^* \mathcal{O}_Q(1) = H_{k+1}$ we obtain:

$$c_{(k+1)r}(E^{(k+1)}) = c_r(E)c_r(E(-1)) \ldots c_r(E(-k)) H_1^r \ldots H_{k+1}^r$$

and so the theorem is proved.

Remark 1 We observe that if the dimension of the locus of k-secants through a generic point is smaller than expected, then the class of the formula has to be zero (see [7] Remark 2.2).

Remark 2 In the case $r = 2$ the theorem has been already proved by Ran in [R]. By the Hartshorne-Serre correspondence every subcanonical subvariety of codimension 2 is a zero locus of a section of a rank 2 vector bundlon $\mathbf{P}^n$; moreover if $n \geq 10$ by Larsen's theorem we have that every subvariety is subcanonical. In this case the formula for $j+1$-secant is true for every subvariety.

4. A new proof of Zak theorem on linear normality

Let X be a r codimensional subvariety of $\mathbf{P}^n$; from Barth theorem we have that if $r \geq n/4$ then $H^{2i}(X, \mathbb{Z}) \simeq \mathbb{Z}$; in particular we can write $c_i(N) = c_i H^i$ with $i = 1 \ldots r$ where $c_i \in \mathbf{Z}$. From now, we consider $c_i(N)$ as a integer.

Lemma 4.1. *Let X be a r codimensional subvariety of $\mathbf{P}^n$. If $n \geq 4r$, then the degree of set of bisecant to X through an external point is*

$$c_r(N)c_r(N(-1)).$$

Proof. Let P be the fixed point, if we project X from P to a generic hyperplane we can use the double point formula [5] to get the set of bisecant to X from P.

$$2\Sigma_2 = f^* f_*[X] - (c(f^* T\mathbf{P}^{n-1}) c(TX)^{-1})_{r-1} \cap [X]$$

and from the exact sequence

$$0 \longrightarrow T_X \longrightarrow T\mathbf{P}^n_{|S} \longrightarrow N_{X, \mathbf{P}^n} \longrightarrow 0$$

we have $c(TX)^{-1} = c(T\mathbf{P}^n)^{-1} c(N)$ and substituting we get

$$2\Sigma_2 = H^{r-1}(d - c_{r-1} + c_{r-2} + \ldots (-1)^i c_i \ldots) = c_r(N(-1))H^{r-1}.$$

From theorem 1.1 and from Lemma we get a different proof of Zak's theorem.

Theorem 4.2 (Zak)**.** *Let X be a r codimensional subvariety of $\mathbf{P}^n$, if $n \geq 4r$, then X is linearly normal.*

Proof. We prove the theorem proceeding by induction on r. If $r = 1$ is trivially true. Now we suppose that it is true for $r-1$. If $c_r(N(-1)) \neq 0$ from theorem 1.1 and from lemma 4.1 we have the thesis. If $c_r(N(-1)) = 0$ from lemma 4.1 we have that there are not bisecant to X through an external point P; if we project X from P to a generic hyperplane, we get a smooth subvariety in $\mathbf{P}^{n-1}$ of codimension $r-1$ that is linearly normal by induction. This is a contradiction.

5. Proof of theorem 1.4

Lemma 5.1. *Let $l, m, p \in \mathbf{N}$ such that $l + p = m$ then we have*

$$\binom{l}{t} = \sum_{i=0}^{k} (-1)^i \binom{m}{t-i} \binom{p-1+i}{i}$$

$$\sum_{1=0}^{t} (-1)^i \binom{n}{t-i} \binom{n+1+i}{i} = (-1)^t$$

Proof. We consider an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where A, B, C are vector spaces with dimension respectively l, m, p. From this exact sequence we obtain other two exact sequences:

(1) $0 \longrightarrow \wedge^t A \longrightarrow \wedge^t B \longrightarrow \wedge^{t-1} B \otimes C \cdots \longrightarrow \wedge^{t-i} \otimes S^i C \longrightarrow \cdots \rightarrow S^n C \longrightarrow 0$

(2) $0 \longrightarrow \wedge^t A \longrightarrow \wedge^{t-1} A \otimes B \cdots \longrightarrow \wedge^{t-i} A \otimes S^i B \rightarrow \cdots \rightarrow S^n B \rightarrow S^n C \rightarrow 0$

considering that $\wedge^t(\mathbf{C}^m) = \binom{m}{t}$ and $S^t(\mathbf{C^n}) = \binom{n-1+t}{t}$ we have

$$\binom{l}{t} = \sum_{1=0}^{t} (-1)^i \binom{m}{t-i} \binom{p-1+i}{i}$$

From (2) if $m = n+1$ and $l = n$ we have:

$$\sum_{1=0}^{t} (-1)^i \binom{n}{t-i} \binom{n+1+i}{i} = (-1)^t$$

Lemma 5.2. *Let X a r codimensional subvariety of $\mathbf{P}^n$ then if $n \geq 4r$ the locus of trisecant is*

$$\Sigma_3 = \frac{1}{2} H^{2r-2} c_r(N(-1)) c_r(N(-2))$$

Proof Göttsche's formula for trisecant through a fixed point is

$$\Sigma_3 = (a) + (b) - (c)$$

where:

$$(a) = H^{2r-2} \left(\frac{n}{2} d^2 - \sum_{k=0}^{n-r} \left(\binom{2n-2r+2}{k} - \binom{n}{k-n+2r-2} \right) \int_X H^k s_{n-r-k}/2 \right)$$

$$(b) = \sum_{k=0}^{2r-2} \sum_{t=0}^{n-1} \binom{n}{t} \binom{n+1}{k-t} \sum_{j=r-t-1}^{2r-2-k} 2^{j+t-r+1} s_j(X) s_{2r-2-k-j}(X) H^k$$

and

$$(c) = \sum_{k=0}^{2r-2} d \binom{n+r}{k} s_{2r-2-k}(X) H^k.$$

We prove the Lemma when r is even (the case r odd is the same). It is well known:

$$s_k = \sum_{i=0}^{n} (-1)^{k+1} H^{k-i} c_i(N) \binom{n+k-i}{k-i}.$$

Let $c_i = c_i(N)$; substituting we have:

$$
\begin{aligned}
(a) \quad = \quad & H^{2r-2}\Big(\frac{n}{2}d^2 - \frac{1}{2}\Big(\sum_{k=0}^{n-r}\binom{2n-2r+2}{k} \cdot \\
& \sum_{i=0}^{r}(-1)^{n-r-k+i}c_i H^{n-r-i}\binom{n+n-r-k-i}{n-r-k-i} + \\
& -\sum_{k=0}^{n-r}\sum_{i=0}^{r}(-1)^{n-r-k+i}\binom{n}{k-n+2r-2}\binom{n+n-r-k-i}{n-r-k-i}\Big)\Big)
\end{aligned}
$$

we put $k' = n-r-k-i$ and so we have:

$$
\begin{aligned}
(a) \quad = \quad & H^{2r-2}\Big(\frac{n}{2}d^2 - \frac{1}{2}\Big(\sum_{i=0}^{r}c_i H^{n-r-i}\sum_{k'=0}^{n-r-i}(-1)^{k'}\binom{n+k'}{k'}\binom{2n-2r+2}{n-r-i-k'} + \\
& -\sum_{k=0}^{r-2-i}(-1)^{k'}\binom{n}{r-2-i-k'}\binom{n+k'}{k'}\Big)\Big)
\end{aligned}
$$

now we can use the Lemma 5.1 and we obtain

$$
(a) = H^{2r-2}\left(\frac{n}{2}d^2 - \frac{1}{2}\left(d^2(n-2r+1) - \sum_{i=0}^{r-1}(-1)^{r-i}c_i H^{n-r-i}\right)\right) \tag{3}
$$

$$
\begin{aligned}
(b) \quad = \quad & \sum_{t=0}^{n-1}\binom{n}{t}\sum_{j=r-t-1}^{2r-2}2^{j+t+1-r}s_j\sum_{k=t}^{2r-2-j}\binom{n+1}{k-t}\sum_{m=0}^{r}(-1)^{2r-2-k-j+m} \cdot \\
& H^{2r-2-j-m}c_m\binom{n+2r-2-k-j-m}{2r-2-k-j-m}
\end{aligned}
$$

If we denote $k' = 2r+2-k-j-m$ we get

$$
\begin{aligned}
(b) \quad = \quad & \sum_{m=0}^{r}c_m\sum_{t=0}^{n-1}\binom{n}{t}\sum_{j=r-t-1}^{2r-2}2^{j+t+1-r}s_j H^{2r-2-j-m} \cdot \\
& \sum_{k'=0}^{2r-2-j-t-m}(-1)^{k'}\binom{n+1}{2r-2-j-m-t-k'}\binom{n+k'}{k'}
\end{aligned}
$$

From Lemma 5.1 we have that the last sum is equal to 1 if $2r-2-j-m-t=0$ and equal to 0 in the other cases; this fact implies also that $j = 2r-2-m-t \geq r-t-1$ and so we obtain $m \leq r-1$.

$$
(b) = \sum_{m=0}^{r-1}c_m\sum_{t=0}^{n-1}\binom{n}{t}2^{r-1-m}s_{2r-2-m-t}H^t
$$

$$
\begin{aligned}
(b) \quad = \quad & \sum_{m=0}^{r-1}c_m 2^{r-1-m}\sum_{t=0}^{n-1}\binom{n}{t}\sum_{i=0}^{r}(-1)^{2r-2-m-t+i} \cdot \\
& H^{2r-2-i-m}c_i\binom{n+2r-2-m-t-i}{2r-2-m-t-i}
\end{aligned}
$$

let $t' = 2r - s - m - t - i$

$$(b) \quad = \quad \sum_{m=0}^{r-1} \sum_{i=0}^{2r-2-m} c_m c_i H^{2r-2-i-m} 2^{r-1-m} \cdot \sum_{t'=0}^{2r-2-m-i} (-1)^{t'} \binom{n+t'}{t'} \binom{n}{2r-2-m-t-i}$$

and again from the Lemma 5.1 we have

$$(b) = \sum_{m=0}^{r-1} \sum_{i=0}^{2r-2-m} (-1)^{m+i} 2^{r-1-m} c_m c_i H^{2r-2-i-m} \tag{4}$$

$$(c) = \sum_{k=0}^{2r-2} d \binom{n+r}{k} \sum_{i=0}^{r} (-1)^{2r-2-k+i} H^{2r-2-i} c_i \binom{n+2r-2-k-i}{2r-2-k-i}$$

let $k' = 2r - 2 - k - i$

$$\begin{aligned} (c) \quad &= \quad \sum_{i=0}^{r} dH^{2r-2-i} c_i \sum_{k'=0}^{2r-2} \binom{n+r}{2r-2-i-k'} (-1)^{k'} \binom{n+k'}{k'} \\ &= \quad \sum_{i=0}^{r} dH^{2r-2-i} c_i \binom{r-1}{2r-2-i} \end{aligned}$$

$$(c) = dc_{r-1} H^{r-1} + d^2 (r-1) H^{2r-2}. \tag{5}$$

Supposing that we are in the range of Barth's theorem, we have that $c_i = c_i H^i$ where $c_i \in \mathbb{Z}$. Finally we get from (3), (4) and (5):

$$\Sigma_3 = (a) + (b) - (c) = \frac{1}{2} H^{2r-2} \sum_{m=0}^{r} \sum_{i=0}^{r} (-1)^{m+i} 2^{r-m} c_m c_i$$

that is

$$\Sigma_3 = H^{2r-2} \frac{1}{2} c_r(N(-1)) c_r(N(-2))$$

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